KATY MCKEOUGH

CHASC ASTRO-STATISTICS

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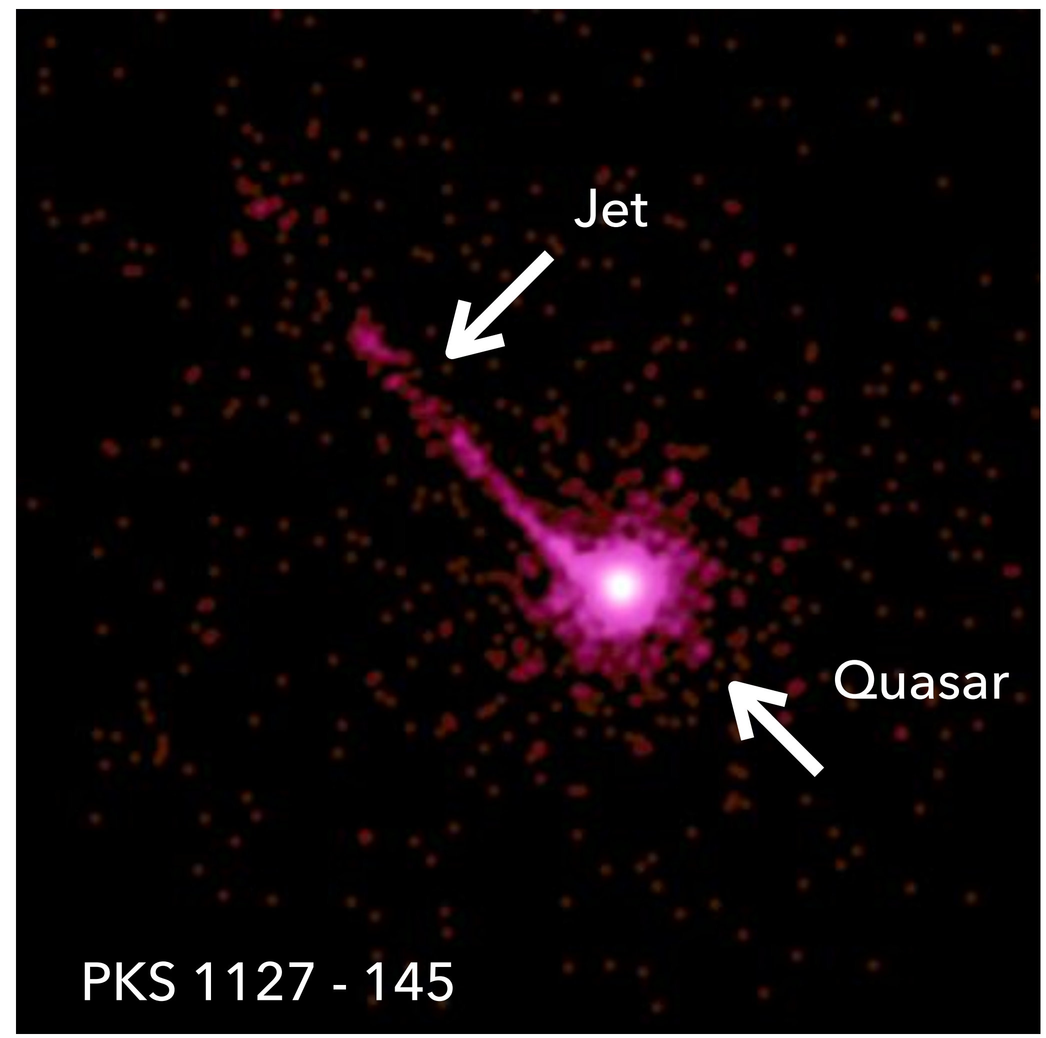
DEFINING REGIONS THAT CONTAIN COMPLEX ASTRONOMICAL STRUCTURES

SCIENTIFIC MOTIVATION

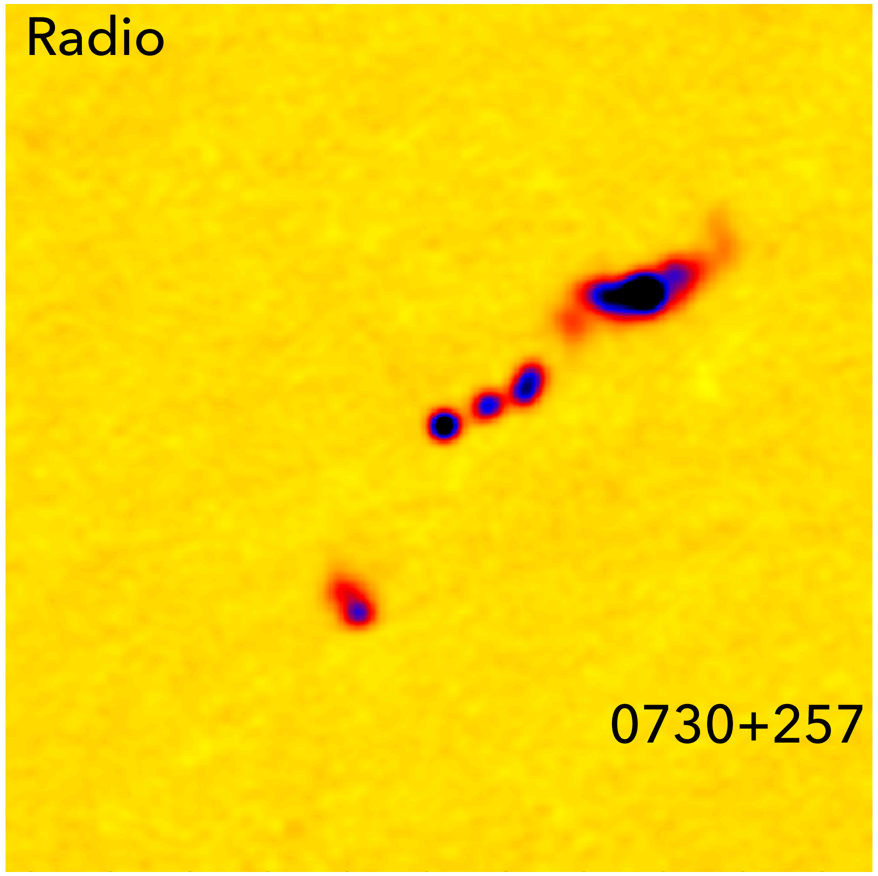
- We are interested in defining an outline around extragalactic jets coming from quasars at high redshift (z>2.1) in X-ray images
- Defining this boundary is important for accurate luminosity and flux calculations.
- Detecting jets is difficult because they are diffuse sources (no edges, or center) and dim compared to the quasar.
- Images of high redshift jets are of low resolution and few X-ray photons

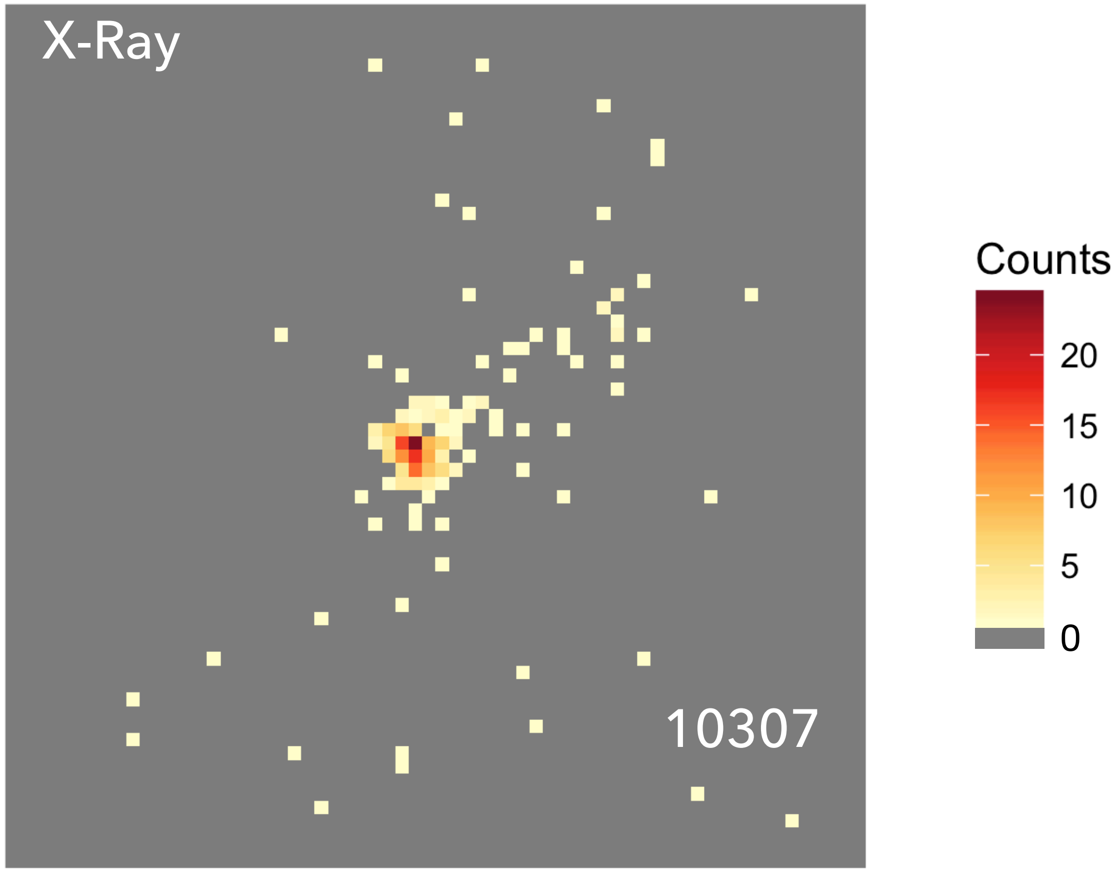


NASA/CXC/A.Siemiginowska(CfA)/J.Bechtold(U.Arizona)

OBSERVATIONAL DATA

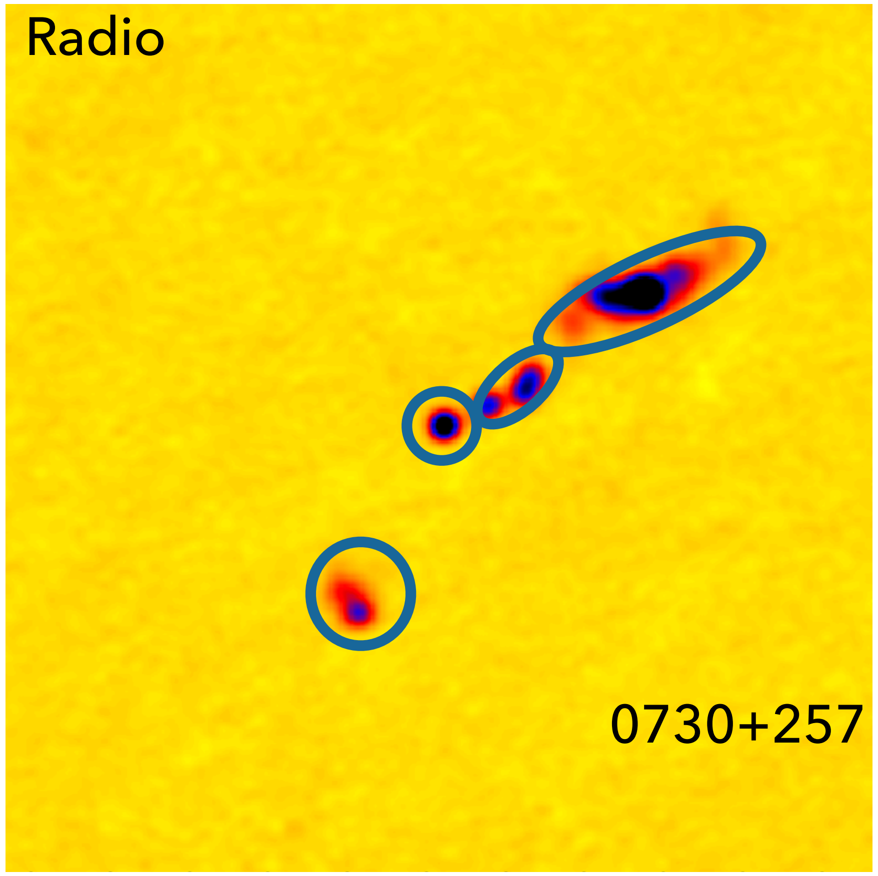
- Chandra X-ray Observatory - ACIS
- 64 x 64 or 128 x 128 pixel image centered on quasar
- High to intermediate redshift (2.10 < z< 4.72)

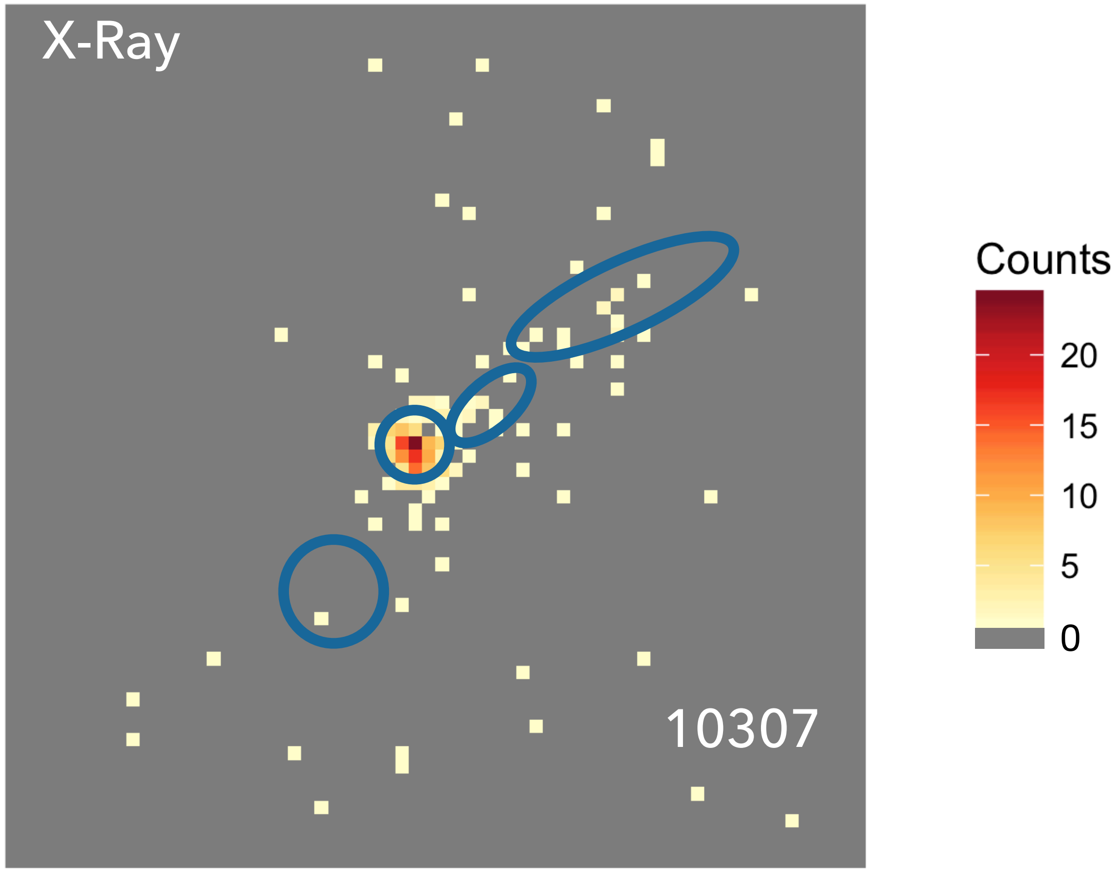




REGION OF INTEREST

- **Region of Interest (ROI)** - region containing the jet or a partition of the jet (e.g. node or lobe)
- Previous work tests whether or not a jet exists in a predefined ROI (McKeough et al. 2016, Stein et al. 2015)





REGION OF INTEREST

- Ability to detect jet is sensitive to fit of ROI
- Issues with previous methods:
  - Region is defined using radio imaging
    - Not always available
    - Not always aligned with X-ray imaging
  - Region definition relies on human interaction
    - Inefficient and source of potential error

GOAL

Define a boundary around the ROI of an irregularly shaped, diffuse source.

Give a measure of uncertainty.

Pre-Process Image

Pixel Assignments

Boundary of ROI

ROADMAP

Pre-Process Image

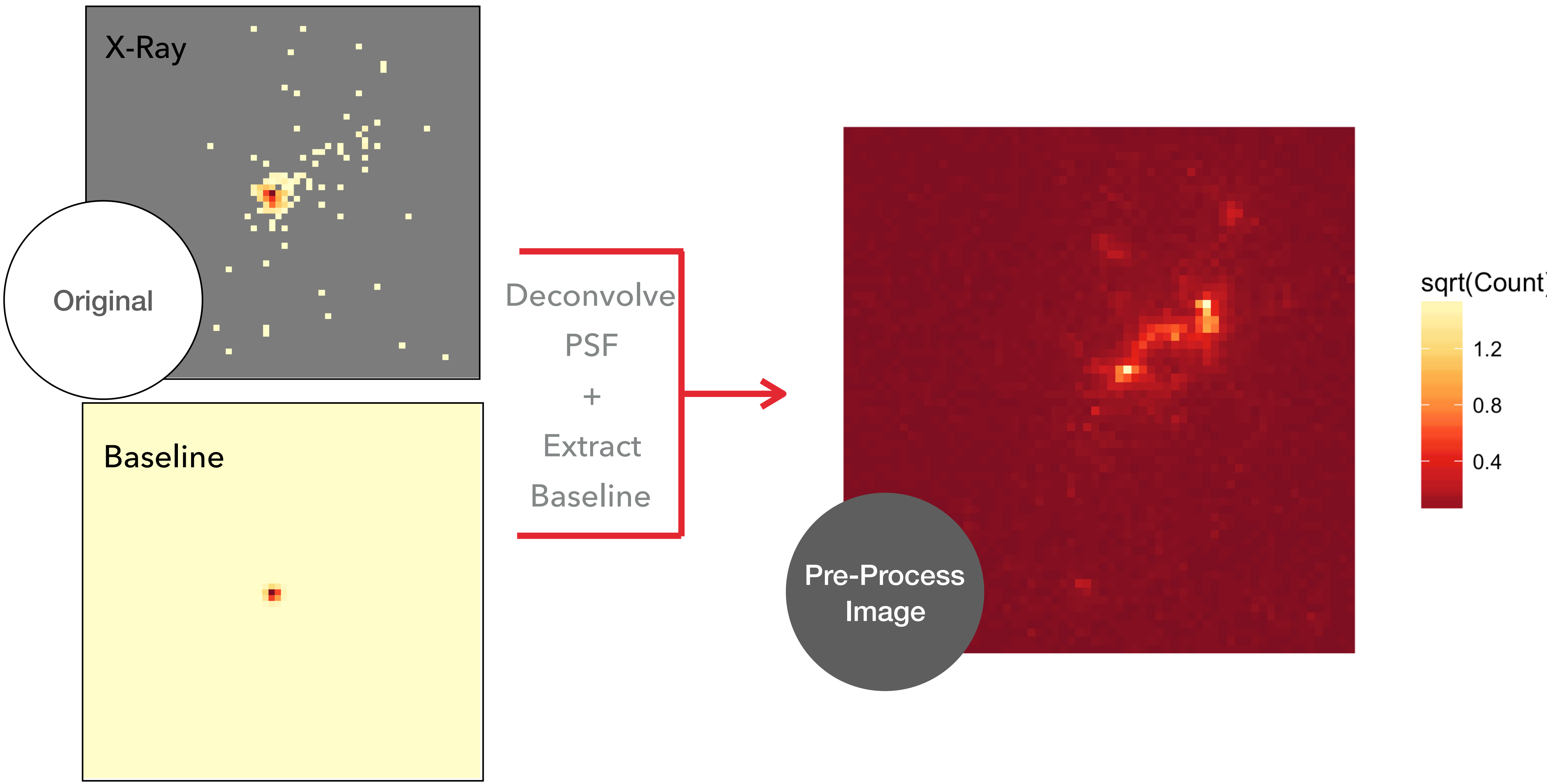
Pixel Assignments

Boundary of ROI

LOW COUNT IMAGE RECONSTRUCTION AND ANALYSIS (LIRA)

- Esch et al (2004), Connors & van Dyk (2007)
- Multi-scale Bayesian method
  - Intensity in "splits" of the image rather than individual pixels
- Removes quasar & deconvolve Point Spread Function (PSF)
- Creates posterior draws for residual pixels as a series of images that capture the emission that is present in excess of the quasar (i.e. the jet)

LOW COUNT IMAGE RECONSTRUCTION AND ANALYSIS (LIRA)



ROADMAP

Pre-Process Image

Pixel Assignments

Boundary of ROI

LIKELIHOOD

$$\sqrt{\tilde{\lambda}_{ij}}|Z, \tau_{\pm}, \sigma^2_{\pm} \sim \text{Normal}(\tau_-, \sigma^2_-)\mathbb{I}_{z_{ij}=-1} + \text{Normal}(\tau_+, \sigma^2_+)\mathbb{I}_{z_{ij}=+1}$$

- We are given observation Y from which we draw the LIRA output: $\tilde{\lambda}|Y$
- We want to assign each pixel to either the background (-1) or the ROI (+1): $z_{ij} = \{-1, +1\}$
- Each pixel assignment will have its own average intensity: τ_-, τ_+
- We suspect the variance of the source will be greater than the background: σ^2_-, σ^2_+

2D ISING PRIOR

$$p(z|\beta) = \frac{\exp(\beta \sum_{ij,i'j' \in |ij-i'j'|=1} z_{ij} z_{i'j'})}{\tilde{Z}(\beta)}$$

- Inverse temperature: β
  - Higher β induces more correlation between pixels
- Partition function: $\tilde{Z}(\beta)$
  - Estimated via Beale (1996) assuming periodic structure
- Commonly used in modeling ferromagnetism.
- Induces spatial correlation; adjacent pixels will tend to have the same assignment.

REMINDER: MODEL SETUP

- **Likelihood:**

$$\sqrt{\tilde{\lambda}_{ij}}|Z, \tau_{\pm}, \sigma^2_{\pm} \sim \text{Normal}(\tau_-, \sigma^2_-)\mathbb{I}_{z_{ij}=-1} + \text{Normal}(\tau_+, \sigma^2_+)\mathbb{I}_{z_{ij}=+1}$$

- **Prior:**

$$p(z|\beta) = \frac{\exp(\beta \sum_{ij,i'j' \in |ij-i'j'|=1} z_{ij} z_{i'j'})}{\tilde{Z}(\beta)}$$

STEP 1 – LIKELIHOOD PARAMETERS

- Draw from posterior directly:
- Priors:

$$\tau_{\pm} \sim \text{Normal}(\mu_0, \sigma^2_{\pm})$$
$$\sigma^2_{\pm} \sim \text{Inv-}\chi^2(\nu_0, \omega^2_0)$$

STEP 2 – TEMPERATURE PARAMETER

- Drawn through Metropolis Hastings
- Prior:

$$\beta \sim \text{Gamma}(a_\beta, b_\beta)$$

STEP 3- ASSIGNMENTS

- A well established way to draw the spin state given a specific temperature is **Swendsen & Wang (1987).**
- The S-W method takes a spin system z|β and induces a bigger system that contains the original *N* spin variables and *M* additional bond variables, denoted by *d*.
- Define joint distribution that couples spins to bonds:

$$p(z, d|\tilde{\lambda}, \tau_{\pm}, \sigma^2_{\pm}, \beta) \propto \prod_{m=1}^{M} g_m(z_m, d_m|\beta) \prod_{ij} f(\tilde{\lambda}_{ij}|z, \tau_{\pm}, \sigma^2_{\pm})$$

  - Marginal distribution of *z* is equal to our posterior.

$$\sum_d p(z, d|\tilde{\lambda}, \tau_{\pm}, \sigma^2_{\pm}, \beta) = p(z|\tilde{\lambda}, \tau_{\pm}, \sigma^2_{\pm}, \beta)$$

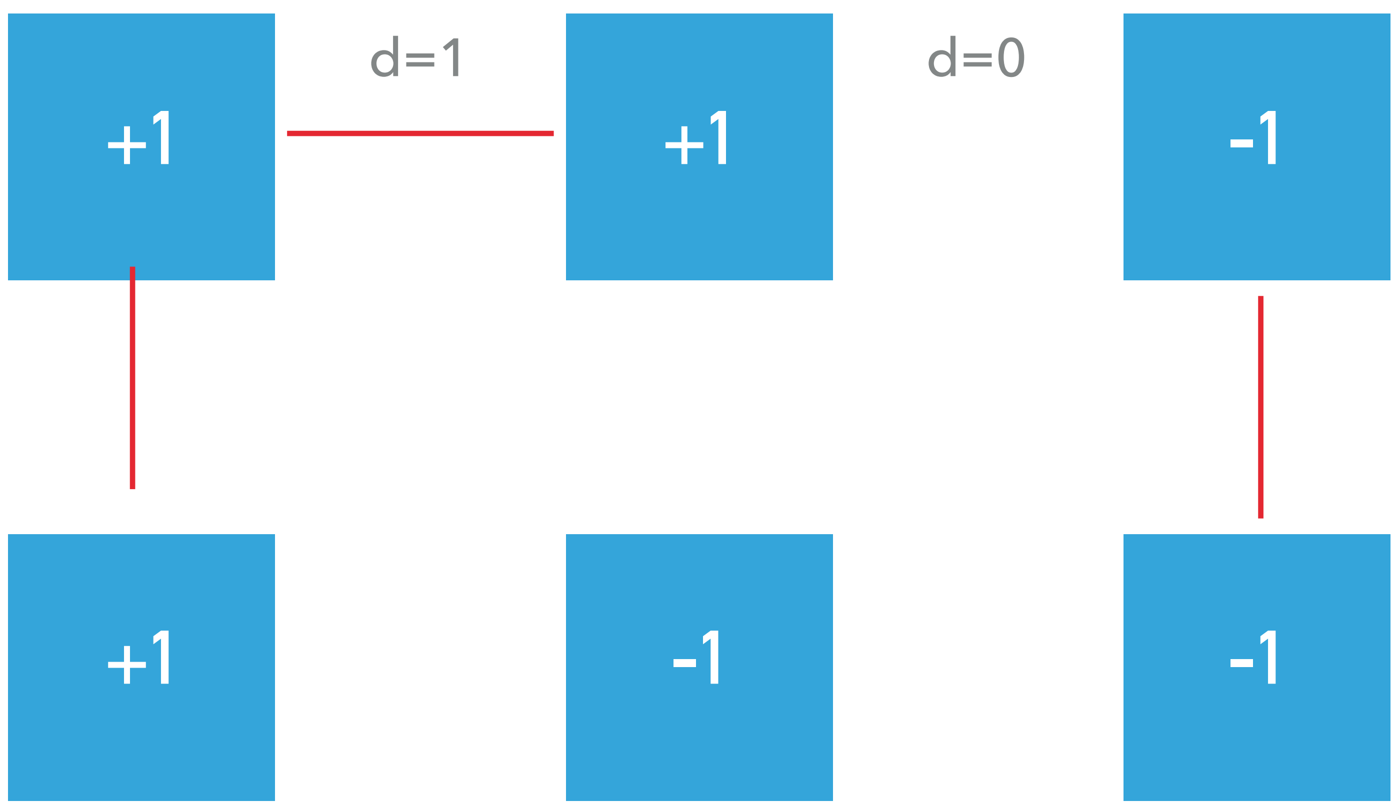
  - Conditional distributions are easy to sample from.

$$p(z|d, \beta -) \qquad p(d|z, \beta -)$$

COUPLING SPINS TO BONDS

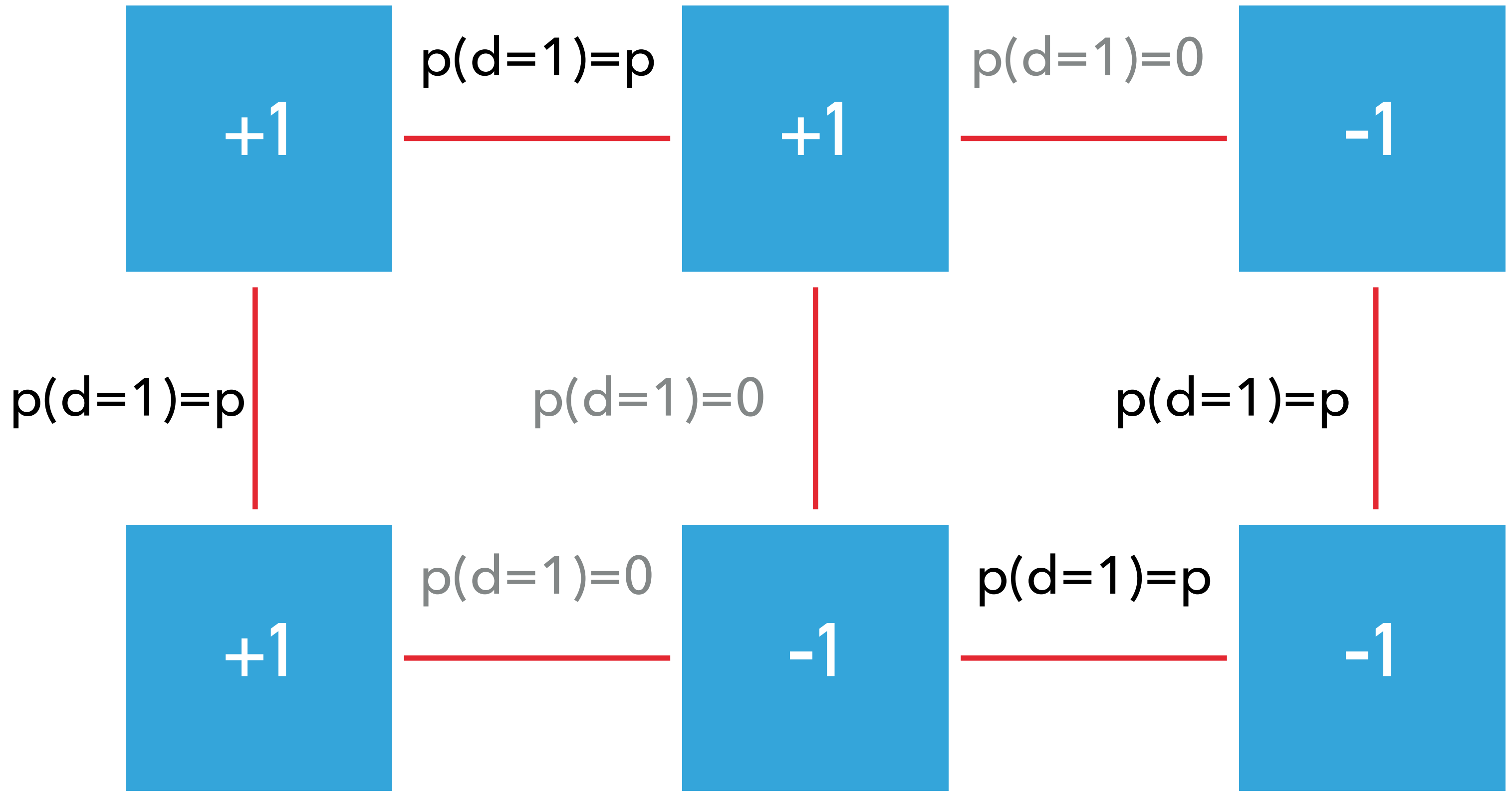
- Bonds can be disconnected (0) or connected (1).

$$d = \{0, 1\}$$



SAMPLING

- Sample from $p(d|z, \beta)$
  - If two spins connected to bond are equal, set the bond d_m equal to 1 with probability *p=1-exp(-2β)* , and 0 otherwise.



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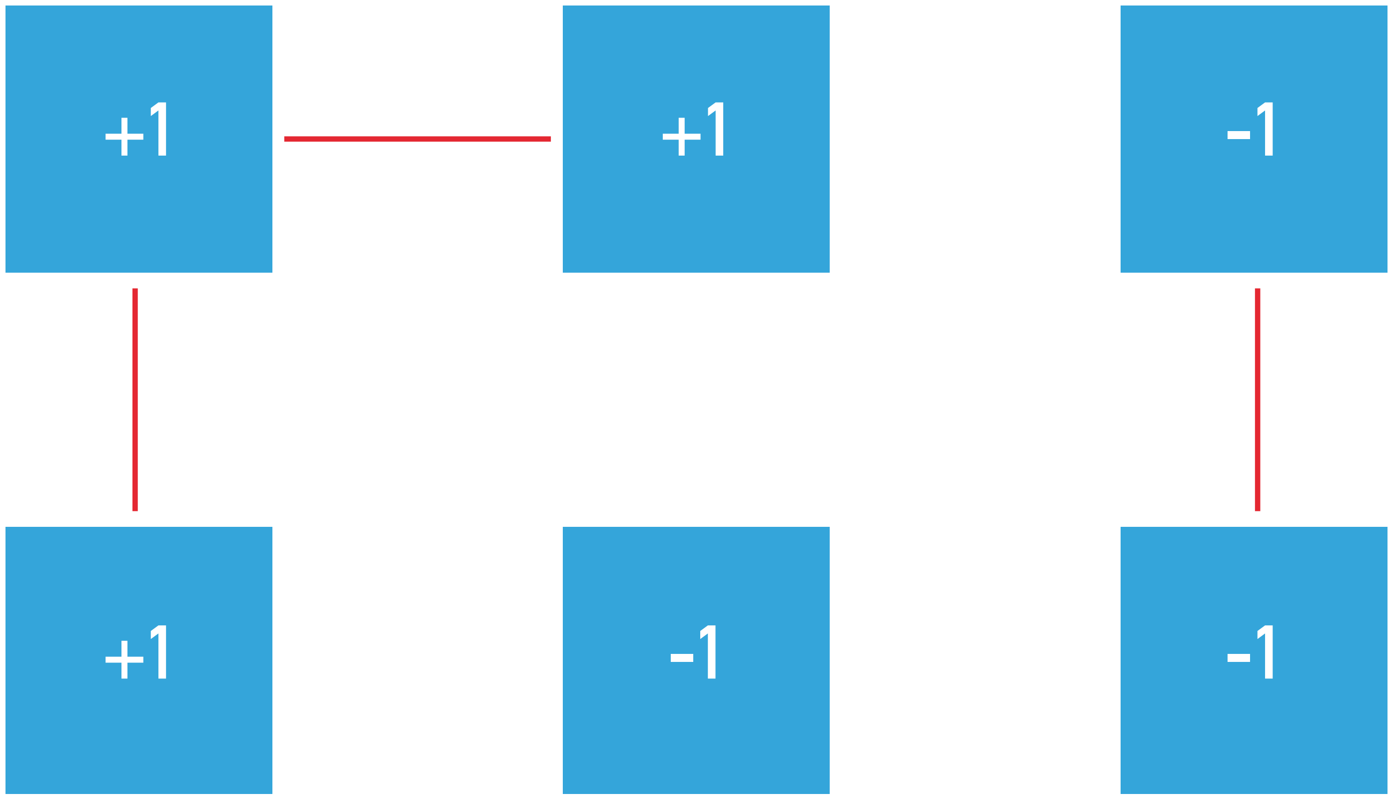


SAMPLING

- Sample from $p(z|d, \beta)$
  - Bonds connect spins into *C* cluster.
  - **Cluster** - all pixels that are connected by a bond d_m=1
  - Each cluster will take spin +1 with probability p_+

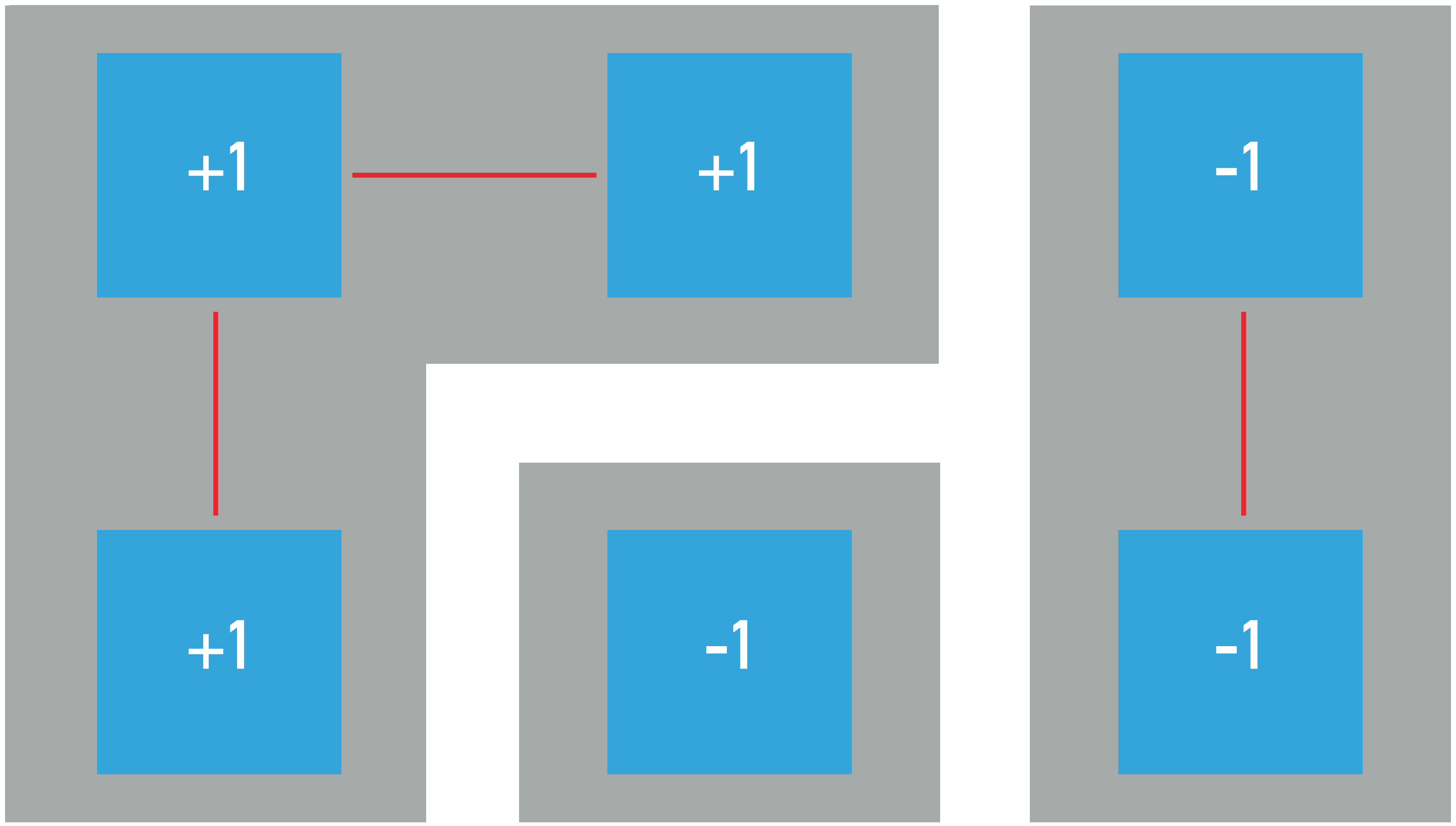
    -1 with probability $p_-=1-p_+$

$$p_{\pm} \propto \prod_{ij \in C} f(\tilde{\lambda}_{ij}|z_{ij} = \pm 1, \tau_{\pm}, \sigma^2_{\pm})$$

SAMPLING

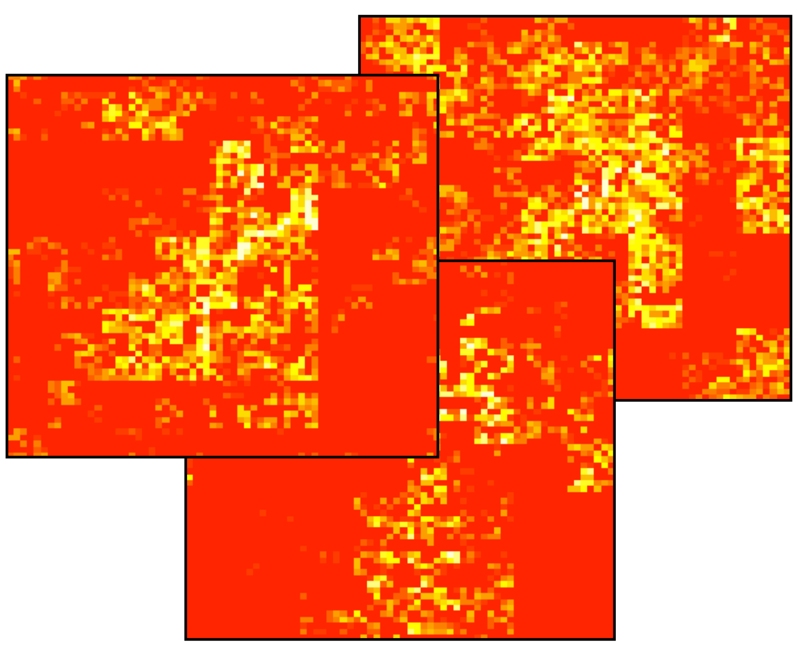
- Sample from $p(z|d, \beta)$

$p(z=+1)=p_+$

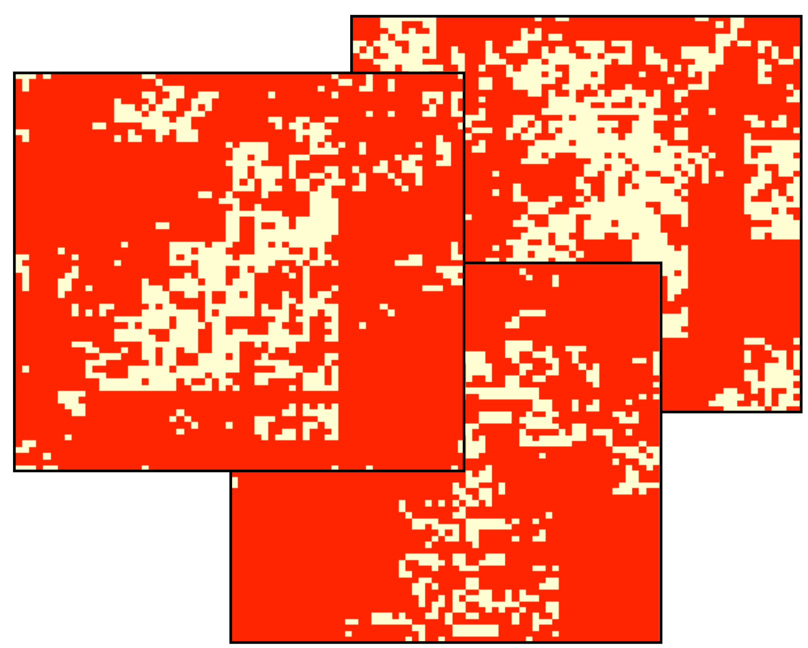


ISING-LIRA ITERATIONS

1. Get many posterior draws from LIRA
2. Apply Ising step to each LIRA draw
3. Average across LIRA-Ising iterations to get probability map.



1



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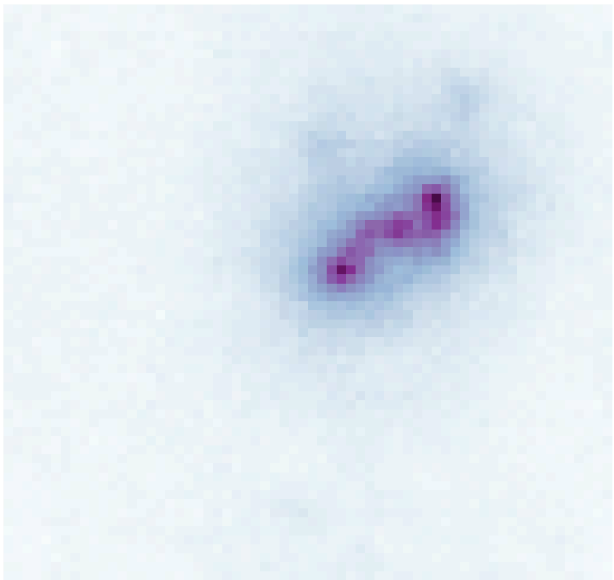
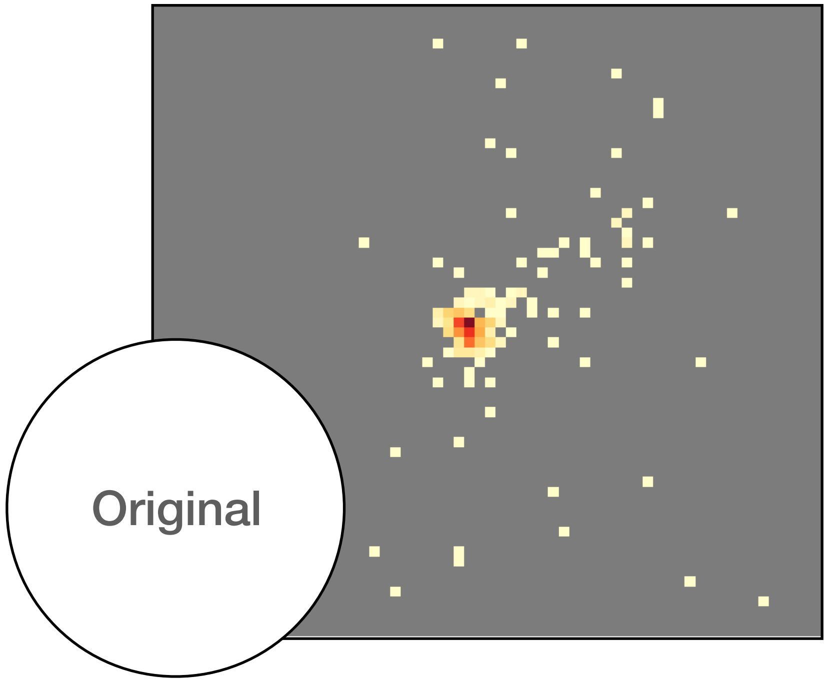
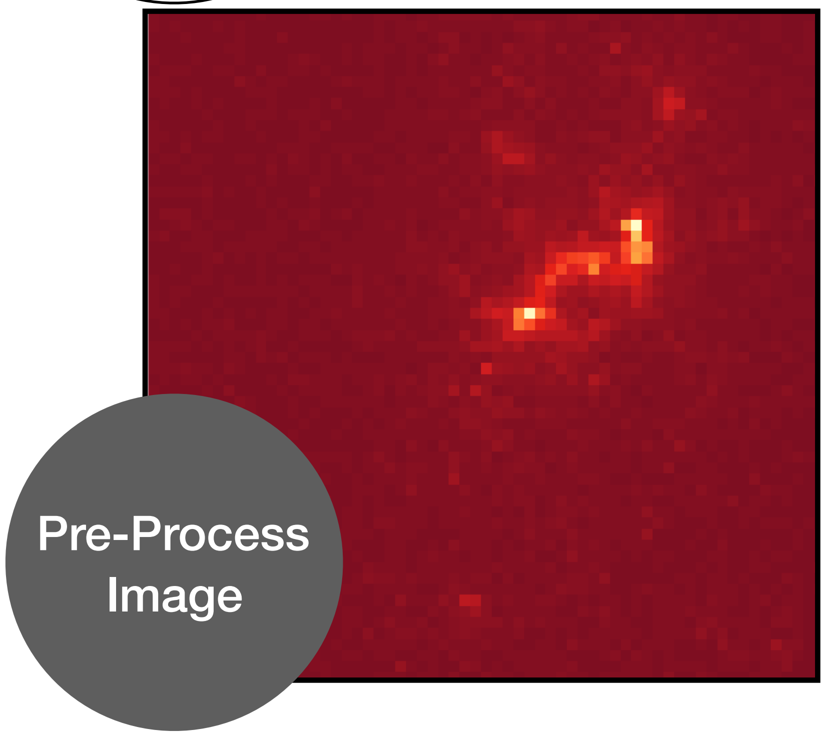


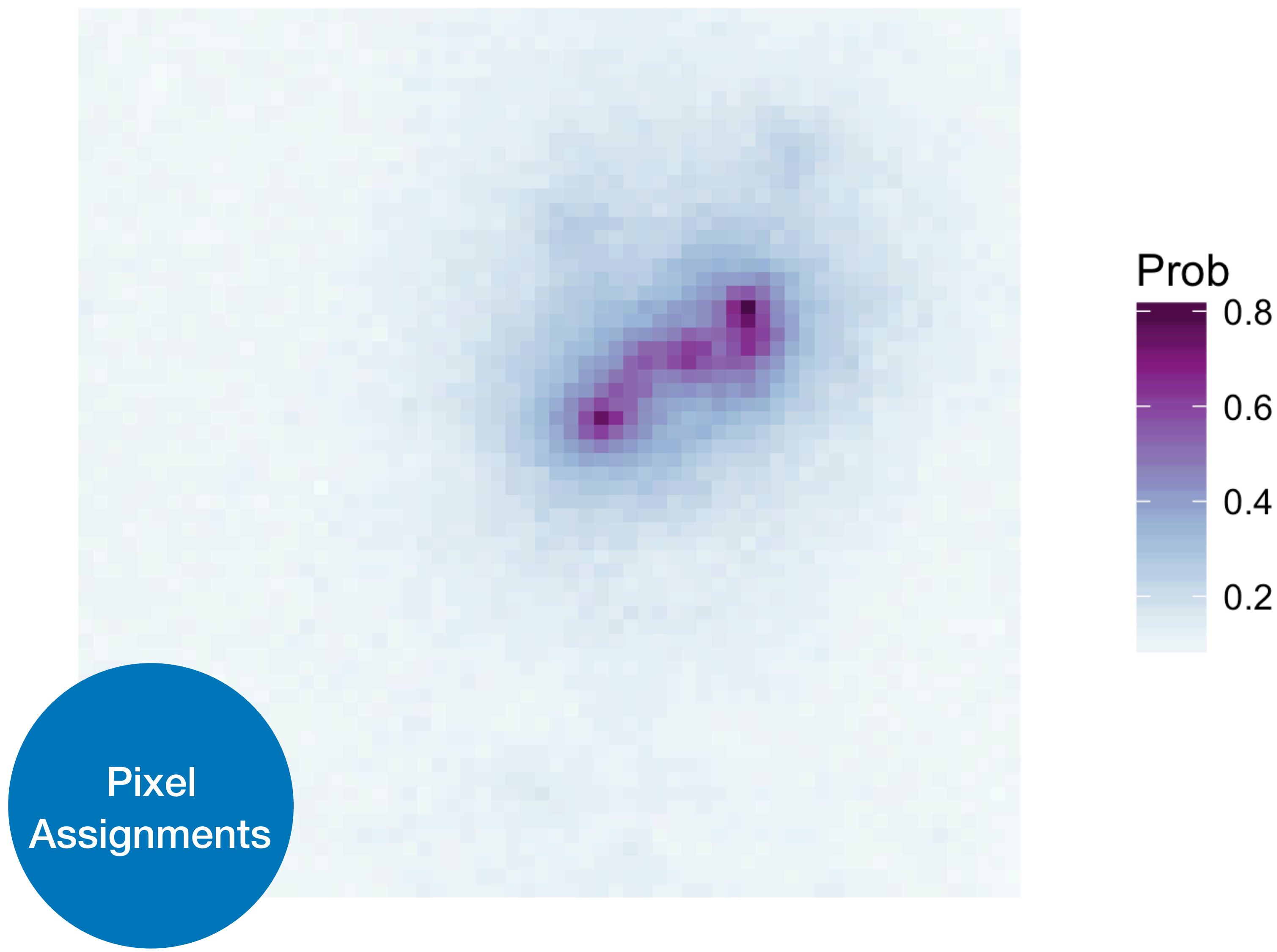
3

PROBABILITY MAP

- Probability each pixel is a member of the ROI:







ROADMAP

Pre-Process Image

Pixel Assignments

Boundary of ROI

OPTIMAL ROI

- Maximize posterior predictive:

$$P(Z|Y) \quad = \quad \int P(Z, \theta, \lambda|Y) d\theta d\lambda$$

OPTIMAL ROI

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$$P(Z|Y) = \int P(Z, \theta, \lambda | Y) d\theta d\lambda$$

- Ideally we could approximate this as:

$$\hat{P}(Z|Y) = \frac{1}{N} \sum_{k=1}^{N} P(Z|\theta^{(k)}, \lambda^{(k)})$$

OPTIMAL ROI

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- Ideally we could approximate this as:

$$\hat{P}(Z|Y) = \frac{1}{N} \sum_{k=1}^{N} P(Z|\theta^{(k)}, \lambda^{(k)})$$

… but this is very difficult.

MAXIMIZE POSTERIOR RATIO

- Compare two different Z states:

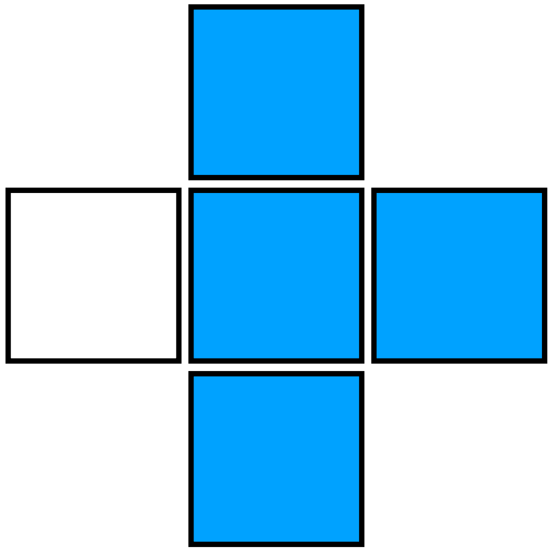
$$\frac{\hat{P}(Z_1|Y)}{\hat{P}(Z_2|Y)} = \frac{\sum_{k=1}^{N} \exp(\log P_k(Z_1))}{\sum_{k=1}^{N} \exp(\log P_k(Z_2))}$$

$$= \sum_{k=1}^{N} w_k \exp(\log \frac{P_k(Z_1)}{P_k(Z_2)})$$

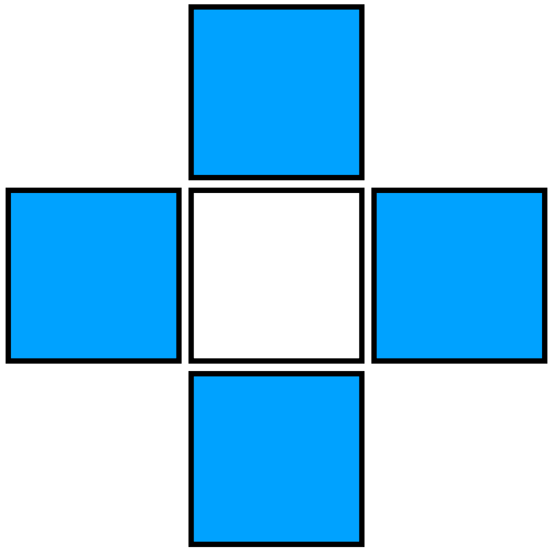
OPTIMIZATION SPACE

- Neighborhood statistic:

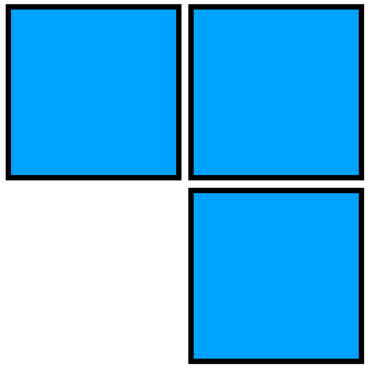
$$N_{ij} = \frac{\sum_{i'j' \in |ij - i'j'| = 1} z_{ij} z_{i'j'}}{\sum_{i'j' \in |ij - i'j'| = 1} |ij - i'j'|}$$



$N_{ij} = 0.75$



$N_{ij} = 0$



$N_{ij} = 1$

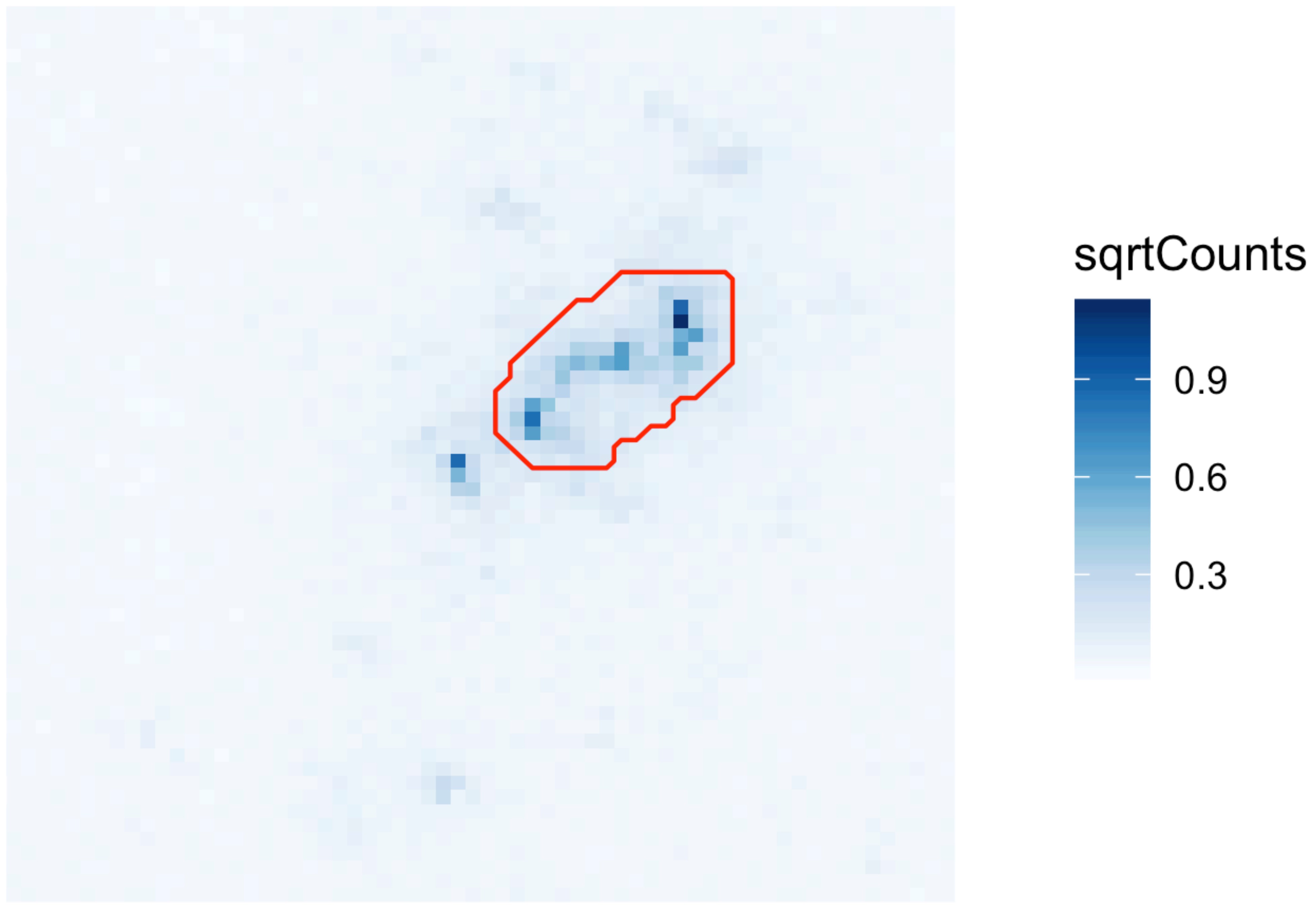
OPTIMIZATION SPACE

- Average N_{ij} across all posterior draws
- Rank N_{ij} from highest to lowest
- Build space to optimize over:
  - For the zip with the highest corresponding N_{ij}, set to 1 and the remainder to -1
  - Repeat including the next highest N_{ij} until all pixels are 1

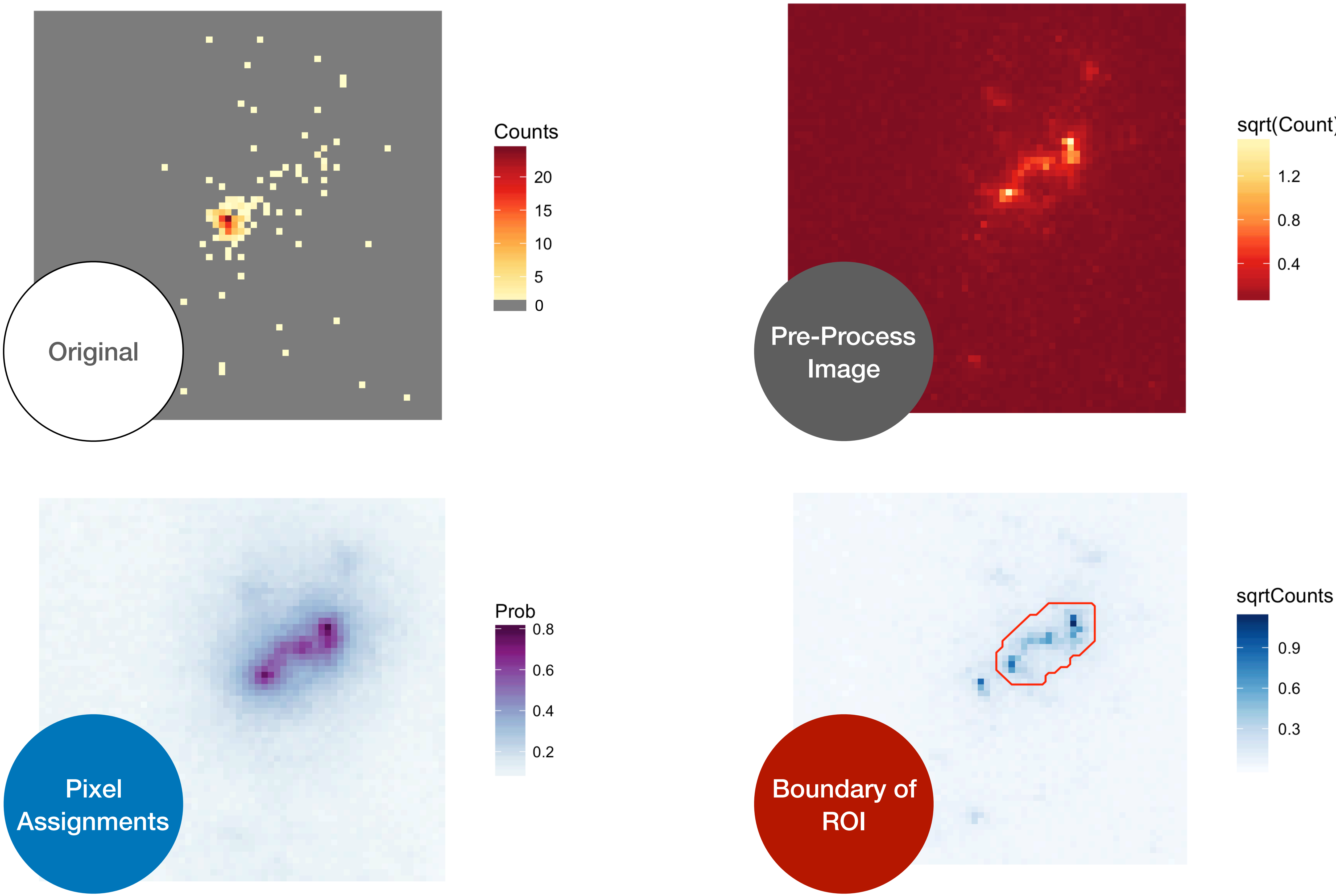
NOTES

- Maximize across all Z created using the neighborhood statistic and all Z drawn from the posterior
- We will always compare the new Z with the Z at the current maximum
- To build a confidence interval take more posterior iterations and repeat the process (TBD)

BOUNDARY OF ROI



RESULTS


Counts
20
15
10
5
0
Original
sqrt(Count)
1.2
0.8
0.4
Pre-Process Image
Prob
0.8
0.6
0.4
0.2
Pixel Assignments
sqrtCounts
0.9
0.6
0.3
Boundary of ROI

Future Work

Pre-Process Image

Pixel Assignments

Boundary of ROI

ADJACENT PIXEL DEFINITION

- Could be modified to the 8 nearest pixels instead of 4.
- Modified to include pixels beyond just the adjacent pixels
- Correlation as a function of distance

POTTS MODEL

- Want to identify multiple partitions of the jet (e.g. nodes)
- Potts is a more generalized version of the Ising model allows for more than two spin assignments:

$$z_{ij} = \{0, 1, 2, 3, \dots\}$$

DIFFERENT LIKELIHOODS

- **Hurdle model** - Account for many of the background pixels in the LIRA output being zero.

CONCLUSION

- LIRA has been successful in analyzing low count images and extracting noisy structure.
  - No way to define a ROI
  - No correlation structure between pixels
- Utilized an Ising distribution and corresponding techniques to create a probabilistic ROI.

Model Compatibility

Pre-Process Image

Pixel Assignments

Boundary of ROI

"IDEAL" MODIFICATION TO LIRA

- Curent LIRA output:

$$P(\tilde{\lambda}|Y)$$

- The missing piece of LIRA is the pixel membership indicator:

$$z_{ij} = \{-1, +1\}$$

- An ideal joint model (denote using subscript $\mathcal{J}$) would infer λ_{ij} and z_{ij} simultaneously

$$P_{\mathcal{J}}(\tilde{\lambda}, z|Y) \propto f(Y|\tilde{\lambda}, z)\pi_{\mathcal{J}}(\tilde{\lambda}, z)$$

OUR APPROACH

- Two-step approach:
  - LIRA "as is" (model S_1)

$$P_{\mathcal{S}_1}(\tilde{\lambda}|Y) \propto f(Y|\tilde{\lambda})\pi_{\mathcal{S}_1}(\tilde{\lambda})$$

  - Ising (model S_2) conditional on ONE draw of from S_1

$$P_{\mathcal{S}_2}(z|\tilde{\lambda}) \propto P_{\mathcal{S}_2}(\tilde{\lambda}|z)\pi_{\mathcal{S}_2}(z)$$

- Combine to get desired model:

$$\begin{aligned} P_{\mathcal{S}}(\tilde{\lambda}, z|Y) &= P_{\mathcal{S}_1}(\tilde{\lambda}|Y)P_{\mathcal{S}_2}(z|\tilde{\lambda}) \\ &\propto f(Y|\tilde{\lambda})\pi_{\mathcal{S}_1}(\tilde{\lambda})\frac{P_{\mathcal{S}_2}(\tilde{\lambda}|z)\pi_{\mathcal{S}_2}(z)}{P_{\mathcal{S}_2}(\tilde{\lambda})} \end{aligned}$$

SUFFICIENT CONDITIONS

$$P_{\mathcal{J}}(\tilde{\lambda}, z|Y) \propto f(Y|\tilde{\lambda}, z)\pi_{\mathcal{J}}(\tilde{\lambda}, z) \quad \Leftrightarrow \quad \begin{aligned} P_{\mathcal{S}}(\tilde{\lambda}, z|Y) &= P_{\mathcal{S}_1}(\tilde{\lambda}|Y)P_{\mathcal{S}_2}(z|\tilde{\lambda}) \\ &\propto f(Y|\tilde{\lambda})\pi_{\mathcal{S}_1}(\tilde{\lambda})\frac{P_{\mathcal{S}_2}(\tilde{\lambda}|z)\pi_{\mathcal{S}_2}(z)}{P_{\mathcal{S}_2}(\tilde{\lambda})} \end{aligned}$$

- Assignment information does not effect distribution of photon counts:

$$f(Y|\tilde{\lambda}) = f(Y|\tilde{\lambda}, z)$$

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- Assignment information does not effect distribution of photon counts:

$$f(Y|\tilde{\lambda}) = f(Y|\tilde{\lambda}, z)$$

- LIRA prior on photon counts is compatible with Ising model prior on assignments:

$$\pi_{\mathcal{S}_1}(\tilde{\lambda}) = \int \pi_{\mathcal{J}}(\tilde{\lambda}, z)dz = \int P_{\mathcal{S}_2}(\tilde{\lambda}|z)\pi_{\mathcal{S}_2}(z)dz$$

HOW FAR OFF ARE WE?

$$P_{\mathcal{J}}(\tilde{\lambda}, z|Y) \propto f(Y|\tilde{\lambda}, z)\pi_{\mathcal{J}}(\tilde{\lambda}, z)$$

$$\begin{aligned} P_{\mathcal{S}}(\tilde{\lambda}, z|Y) &= P_{\mathcal{S}_1}(\tilde{\lambda}|Y)P_{\mathcal{S}_2}(z|\tilde{\lambda}) \\ &\propto f(Y|\tilde{\lambda})\pi_{\mathcal{S}_1}(\tilde{\lambda})\frac{P_{\mathcal{S}_2}(\tilde{\lambda}|z)\pi_{\mathcal{S}_2}(z)}{P_{\mathcal{S}_2}(\tilde{\lambda})} \end{aligned}$$

- Inference for λ is equivalent:

$$P_{\mathcal{J}}(\lambda|Y) \propto f(Y|\lambda)\int \pi_{\mathcal{J}}(\lambda, z)dz = f(Y|\lambda)\pi_{S_1}(\lambda) \propto P_{S}(\lambda|Y)dz$$

HOW FAR OFF ARE WE?

$$P_{\mathcal{J}}(\tilde{\lambda}, z|Y) \propto f(Y|\tilde{\lambda}, z)\pi_{\mathcal{J}}(\tilde{\lambda}, z)$$

$$\begin{aligned} P_{\mathcal{S}}(\tilde{\lambda}, z|Y) &= P_{\mathcal{S}_1}(\tilde{\lambda}|Y)P_{\mathcal{S}_2}(z|\tilde{\lambda}) \\ &\propto f(Y|\tilde{\lambda})\pi_{\mathcal{S}_1}(\tilde{\lambda})\frac{P_{\mathcal{S}_2}(\tilde{\lambda}|z)\pi_{\mathcal{S}_2}(z)}{P_{\mathcal{S}_2}(\tilde{\lambda})} \end{aligned}$$

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- Posterior inference is bounded by the prior divergence (which can be calculated)

$$D_{KL}(P_{\mathcal{J}}(\lambda, z|Y), P_{\mathcal{S}}(\lambda, z|Y)) = \int P_{\mathcal{J}}(\lambda|Y)D_{KL}(P_{\mathcal{J}}(z|\lambda), P_{\mathcal{S}}(z|\lambda))d\lambda$$

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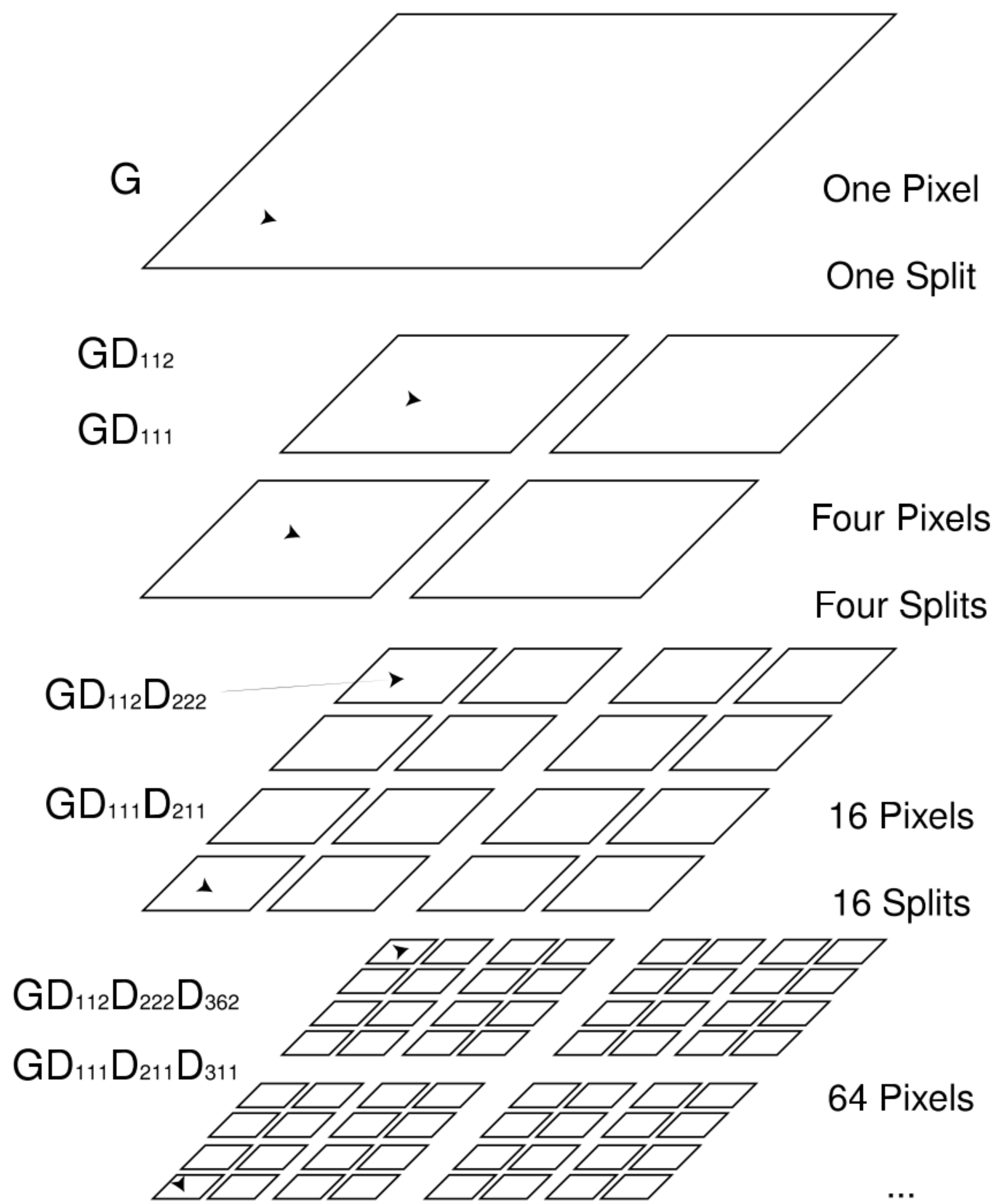
LIRA

MULTI-SCALE IMAGE REPRESENTATION

- Stores total intensities and series of four way split proportions such that the product recovers original pixel intensities
- Pixel Intensity $\Lambda = \{\Lambda_i, I = 1 \ldots N\}$
- Splits $D_{k,l_{k(i)},m_{k(i)}}$
  - Split proportion at scale *k* corresponding to group *i*

$$\Lambda_i = G \prod_{k=1}^{K} D_{k,l_{k(i)},m_{k(i)}}$$

MULTI-SCALE IMAGE REPRESENTATION



LIKELIHOOD

- Probability photon originating in pixel *i*, is observed in pixel *j* *(PSF)*: $P_i = \{P_{ij}, j = 1, \ldots N\}$
- Observed pixel counts: $Y = \{Y_i, i = 1, \ldots N\}$
- Distribution of *Y*:

$$Y_j|\Lambda, \Lambda^B \overset{\mathrm{d}}{\sim} \text{Poisson}\left[\left(\sum_{i \in \mathcal{I}} P_{ij}\Lambda_i\right) + \Lambda_j^B\right]$$

- Suppress background to obtain likelihood:

$$L(\Lambda,\ \Lambda^B|\boldsymbol{Y}) \equiv L(\Lambda|\boldsymbol{Y}) \propto \prod_{j \in \mathcal{I}} p(Y_j|\Lambda)$$

PRIOR

- Prior on total intensity:

$$G \sim \text{Gamma}(\gamma_0, \gamma_1)$$

- Prior on splits:

$$\boldsymbol{D}_{kl} \equiv \{D_{klm},\ m = 1, \ldots, 4\} \overset{\text{d}}{\sim} \text{Dirichlet}(\alpha_k,\ \alpha_k,\ \alpha_k,\ \alpha_k)$$
$$k = 1, \ldots, K, \quad l = 1, \ldots, 4^{k-1}$$

- Hyperprior favors smoother image:

$$p(\alpha_k) \propto \exp(-\delta\alpha^3/3)$$

CYCLE SPINNING

- Multiscale format produces checkerboard-like patterns
- Solution:
  - Shift center of image randomly before making splits
  - Splits wrap around edges of image to induce translation invariance

SWENDSEN-WANG

COUPLING SPINS TO BONDS

- Factor coupling bonds and spins is:

$$g_m(z_m, d_m) = \left\{ \begin{array}{ccccc} & \multicolumn{2}{c}{d_m = 0} & \multicolumn{2}{c}{d_m = 1} \\ & z_{i'j'} = -1 & z_{i'j'} = +1 & z_{i'j'} = -1 & z_{i'j'} = +1 \\ z_{ij} = -1 & e^{-\beta} & e^{-\beta} & e^{\beta} - e^{-\beta} & 0 \\ z_{ij} = +1 & e^{-\beta} & e^{-\beta} & 0 & e^{\beta} - e^{-\beta} \end{array} \right.$$

- Rescale by constant factor: $p = 1 - e^{-2\beta}$

$$\tilde{g}_m(z_m, d_m) = \left\{ \begin{array}{ccccc} & \multicolumn{2}{c}{d_m = 0} & \multicolumn{2}{c}{d_m = 1} \\ & z_{i'j'} = -1 & z_{i'j'} = +1 & z_{i'j'} = -1 & z_{i'j'} = +1 \\ z_{ij} = -1 & 1 - p & 1 - p & p & 0 \\ z_{ij} = +1 & 1 - p & 1 - p & 0 & p \end{array} \right.$$