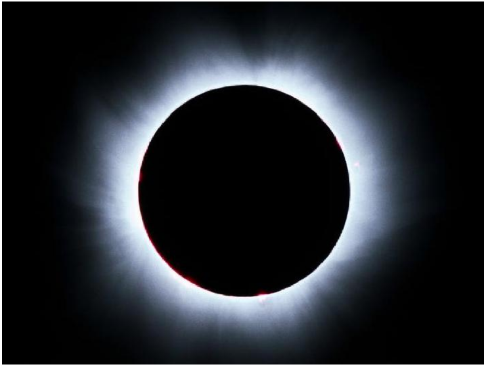
Part II

Bayesian DEM Reconstruction

Stellar Corona



Definition

Corona

- The outermost layer of a stellar atmosphere.

⇐ Sun's corona during a total solar eclipse. (08/11/1999)

Differential Emission Measure and Elemental Abundances

Definition

- Differential Emission Measure (DEM) (μ) : The distribution of the amount emission at different temperatures from a stellar corona.
- Elemental Abundances (γ_k) : The fractions of each element compared to hydrogen relative to solar abundances.

What can we learn?

- DEM summarizes the temperature structure of corona.
- It provides how the corona is cooled, which then feeds back into the physical structure of the corona.
- The elemental composition of a star is an important determinant of its physical structure, and a tracer of its evolutionary state.

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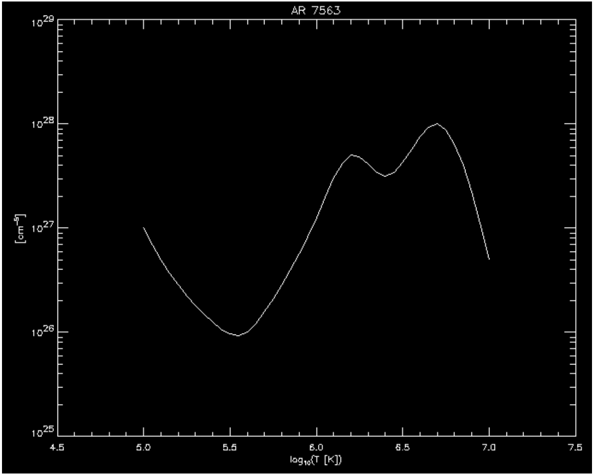
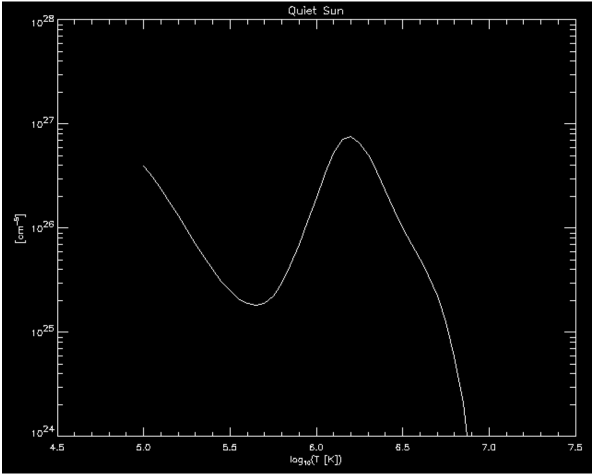
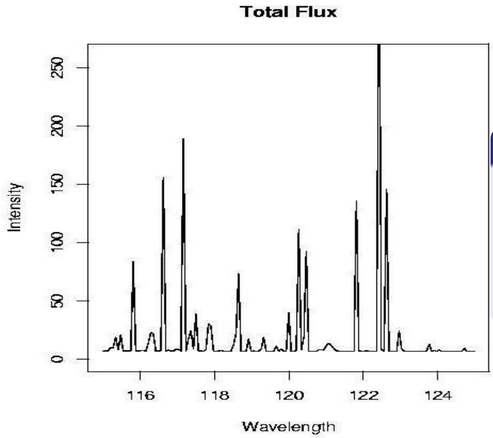
The Solar DEM in an Active Region



The Solar DEM in an Quiet Region



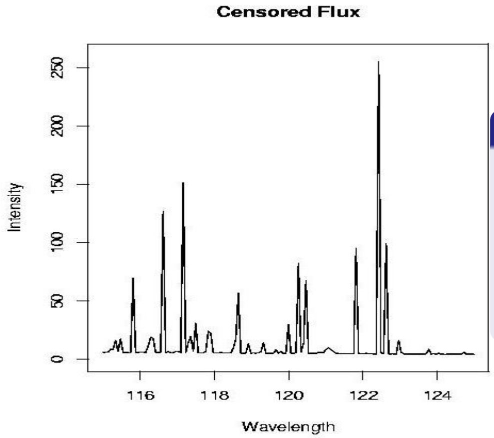
Total Flux : We wish we could observe this.



Total Flux

- Source model energy spectrum - Ideal data.
- The mixture of the continuum and emission lines.

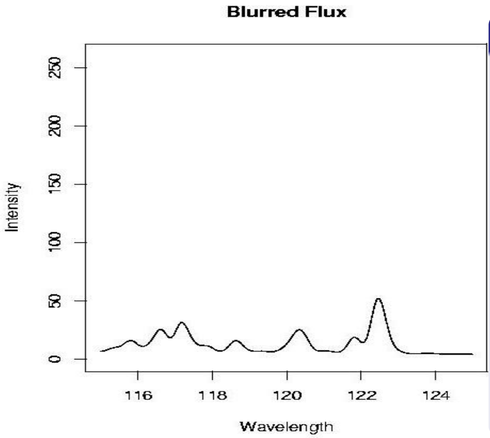
Stochastic Censoring



Effective Area

- A photon has a certain energy dependent probability of being recorded by the detector.
- This relative efficiency is called *effective area*.

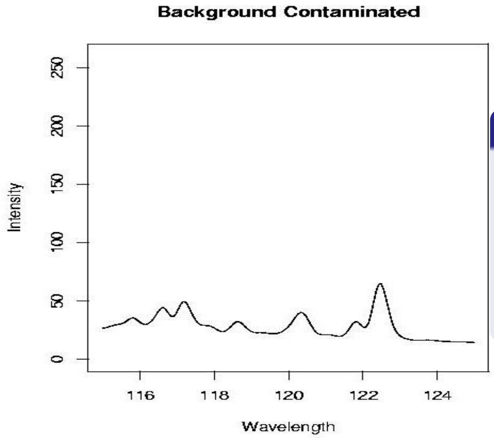
Measurement Errors - Blurring



Blurring

- Mirrors do not focus perfectly.
- The line-spread function characterizes the probability distribution of a photon's recorded energy location relative to its true energy.
- The shape of the distribution are approximated by *t*-distribution or Gaussian distribution.

Background Contamination



Background Contamination

- Photon counts come from other celestial objects that are near the line of sight of the the source of interest. (background contamination)

Differential Emission Measure and Elemental Abundances

Emission line intensity originating from element k:

$$
\begin{aligned}
\lambda_l^{k,L} &= A \int \gamma_k G_l^{k,L}(T)\mathrm{DEM}(T) d \log T \\
&\approx A(\Delta \log T) \sum_{i=1}^{M} \gamma_k \mathbf{G}_{lt}^{k,L} \mu_t, \quad \text{or equivalently} \\
\boldsymbol{\lambda}^{k,L} &= A(\Delta \log T)\gamma_k \mathbf{G}^{k,L}\boldsymbol{\mu} \propto \gamma_k \mathbf{G}^{k,L}\boldsymbol{\mu},
\end{aligned}
$$

- We are interested in γ_k and $\boldsymbol{\mu}$.
- $\mathbf{G}^{k,L}$ is known. (What is $\mathbf{G}^{k,L}$?)

Emissivity Matrix

- They are known from ATOMDB v1.3.
- There are 28 such matrices.
- Big matrix (for example, for chandra data set, 2,160$\times$64)
- Conditional distribution of energy at a given temperature.
- Each column corresponds to the emissivity at a given temperature.
- In continuum emissivity matrix, each row corresponds to a energy bin.
- In line emissivity matrix, each row corresponds to a line location (prone to error).

Differential Emission Measure and Elemental Abundances

(Let's just say...) the expected photon counts at energy bin j, λ_j

$$\begin{aligned}\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_J)' &= \sum_{k=1}^{K} \{\boldsymbol{\lambda}^{C,k} + \mathrm{binning}(\boldsymbol{\lambda}^{L,k})\} \\ &\propto \left(\sum_{k=1}^{K} \gamma_k \{\mathbf{G}^{C,k} + \mathrm{binning}(\mathbf{G}^{L,k})\} \right) \boldsymbol{\mu},\end{aligned}$$

where K is the total number of elements i.e. $K = 14$.

Spectral Model

The observed counts at channel i with background contamination follow independent Poisson variables with intensity

$$\xi_i = \sum_{j=1}^{J} M_{ij}\lambda_j d_j + \lambda_i^B, \quad i = 1, \ldots, I,$$

$$\boldsymbol{\xi} \propto \mathbf{MD}\left(\sum_{k=1}^{K} \gamma_k \{\mathbf{G}^{C,k} + \text{binning}(\mathbf{G}^{L,k})\}\right)\boldsymbol{\mu} + \boldsymbol{\lambda}^B.$$

- d_j : The probability that an X-ray is not refracted off the detector bin j. $\mathbf{D} = \text{diag}(d_j)$.
- M_{ij} : The probability that a photon that arrives with energy corresponding to bin j is recorded in detector channel i.

Conditional Augmentation

We renormalize $\mathbf{G}_{\text{total}}$ to reduce the counts attributed to the censored photons.

$$\mathbf{G}_{\text{total}} = \left(\sum_{k=1}^{K} \gamma_k \{ \mathbf{G}^{C,k} + \text{binning}(\mathbf{G}^{L,k}) \} \right),$$

$$\textbf{norm} = \max_{t=1,\dots,M} \left\{ \sum_{j=1}^{J} \mathbf{G}_{\text{total}\,j,t} \right\}, \mathbf{G}^{\star} = \frac{1}{\textbf{norm}} \mathbf{G}_{\text{total}}$$

Conditional Augmentation - Toy Example

1.

$$\begin{pmatrix}13\\14\\19\end{pmatrix} = \begin{pmatrix}0.10 & 0.15\\0.20 & 0.10\\0.10 & 0.25\end{pmatrix}\begin{pmatrix}??\\??\end{pmatrix}$$

Solve this via EM.

2.

$$\begin{pmatrix}13\\14\\19\end{pmatrix} = {\color{red}0.5}\begin{pmatrix}0.20 & 0.30\\0.40 & 0.20\\0.20 & 0.50\end{pmatrix}\begin{pmatrix}??\\??\end{pmatrix}$$

Conditional Augmentation - Toy Example

1.

$$\begin{pmatrix} 13 \\ 14 \\ 19 \end{pmatrix} = \begin{pmatrix} 0.10 & 0.15 \\ 0.20 & 0.10 \\ 0.10 & 0.25 \end{pmatrix} \begin{pmatrix} 40 \\ 60 \end{pmatrix}$$

Current Guess? $\mu_1 = 40, \mu_2 = 60$

2.

$$\begin{pmatrix} 13 \\ 14 \\ 19 \end{pmatrix} = 0.5 \begin{pmatrix} 0.20 & 0.30 \\ 0.40 & 0.20 \\ 0.20 & 0.50 \end{pmatrix} \begin{pmatrix} 40 \\ 60 \end{pmatrix}$$

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- Multinomial Split!

$$\begin{pmatrix}13\\14\\19\\\hline 46\end{pmatrix} = \begin{pmatrix}4+9\\8+16\\4+15\\\hline 16+30\end{pmatrix}$$

- Restore censored counts:
  1. $\mu_1^{new} = 16 + (1-0.4)40,$
     $\mu_2^{new} = 30 + (1-0.5)60.$
  2. $0.5\mu_1^{new} = 16 + (1-0.8)20,$
     $0.5\mu_2^{new} = 30 + (1-1.0)30.$

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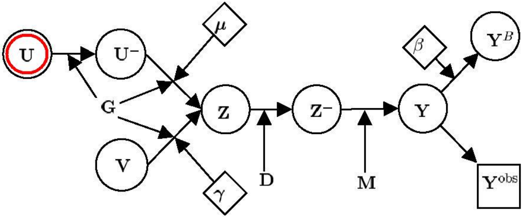
- New estimates: $\mu_1 = 40, \mu_2 = 60$.
- Proportion of missing data :
  1. $24/40 = 60\%$, $30/60 = 50\%$.
  2. $4/20 = 20\%$, $0/30 = 0\%$

Hierarchical Missing Data Structuring



- U_t : Ideal photon count in temperature bin t

  ↓ *Stochastic Censoring*
- U_t^- : Stochastically censored photon count at temperature bin t
- V_k : Count of photons originating from element k

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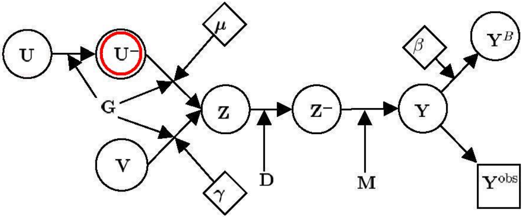

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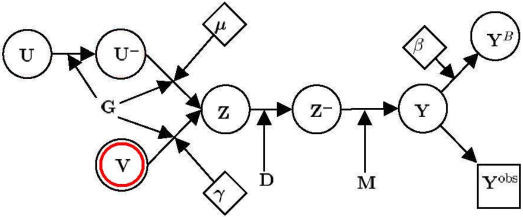

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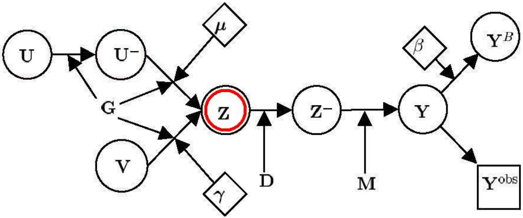

- Z_j : Ideal bin count at energy bin j

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  ↓ *Line Spread Function*

- Y_i : Source count at energy channel i

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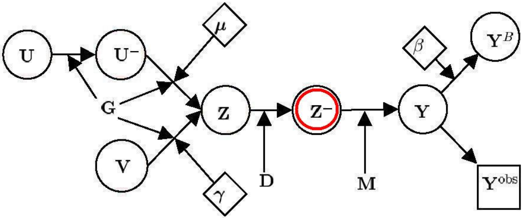

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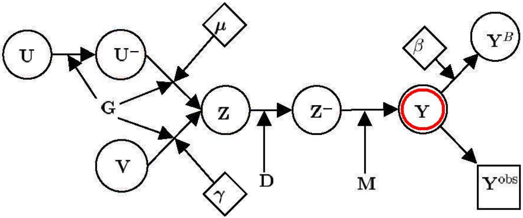

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Hierarchical Missing Data Structuring

$$p(\gamma, \boldsymbol{\lambda}, \beta, \mathbf{V}, \mathbf{U}, \mathbf{U}^-, \mathbf{Z}, \mathbf{Z}^-, \mathbf{Y} | \mathbf{Y}^{\rm obs}) \propto p(\gamma, \boldsymbol{\lambda}, \beta) p(\mathbf{V}|\gamma, \boldsymbol{\lambda}) p(\mathbf{U}|\gamma, \boldsymbol{\lambda}) p(\mathbf{U}^-|\mathbf{U}) p(\mathbf{Z}|\mathbf{U}^-) \times p(\mathbf{Z}^-|\mathbf{Z}) p(\mathbf{Y}|\mathbf{Z}^-) p(\mathbf{Y}^{\rm obs}|\mathbf{Y}, \beta)$$

For example,

- $$Z_j^- | Z_j, \theta \sim \textbf{Binomial}(Z_j, d_j).$$

- $$\mathbf{Z}|\mathbf{U}^-, \theta \sim \sum_t \textbf{Multinomial}\left(U_t^-, \frac{\mathbf{G}^\star_{\bullet t}}{\sum_t \mathbf{G}^\star_{\bullet t}}\right),$$

  where $\mathbf{G}^\star_{\bullet t}$ is the t-th column of $\mathbf{G}^\star$.

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Missing Data Sampling Conditional on Parameters

1. Independently separate the background counts from the observed counts,

$$Y_i^B | Y_i^{\text{obs}}, \theta \sim \textbf{Binomial}(Y_i^{\text{obs}}, \lambda_i^B / \xi(\theta)), \; i = 1, \ldots I,$$

2. Restore the blurred photons,

$$\mathbf{Z}^- | \mathbf{Y}, \theta \sim \sum_{i=1}^{I} \textbf{Multinomial}\left(Y_i, \frac{(M_{1i} d_i \lambda_i, \ldots, M_{Ji} d_J \lambda_J)'}{\sum_j M_{ji} d_j \lambda_j} \right)$$

3. Independently restore the absorbed counts due to the effective area,

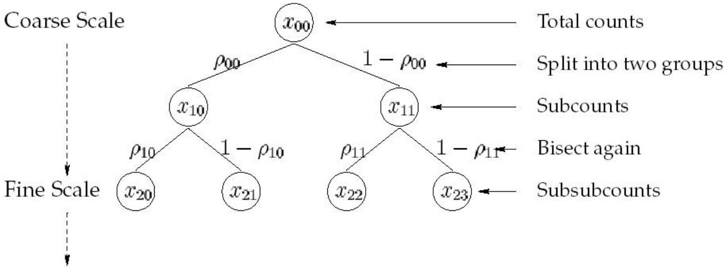
$$Z_j | Z_j^-, \theta \sim Z_j^- + \textbf{Poisson}((1 - d_j)\lambda_j), j = 1, \ldots J.$$

4. Restore $\mathbf{U}^-$ given $\mathbf{Z}, \theta$.

$$\mathbf{U}^- | \mathbf{Z}, \theta \sim \sum_{j=1}^{J} \textbf{Multinomial}\left(Z_j, \frac{\mathbf{G}^\star_{j\bullet} \cdot \boldsymbol{\mu}}{\sum \mathbf{G}^\star_{j\bullet} \cdot \boldsymbol{\mu}} \right),$$

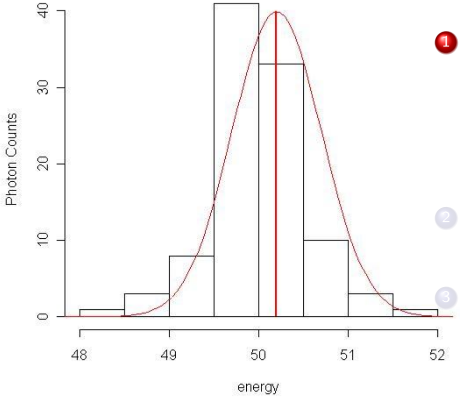
where $\mathbf{G}^\star_{j\bullet}$ is a j-th row vector.

Multi-Scale Smoothing



- The Poisson intensity of a "parent" node is a sum of the Poisson intensity of the two "child" nodes.
- The smoothness of the intensities is controlled by the splitting factors $\{\rho_{r,k}\}$.

Atomic Data Errors - Emission Line Location Correction



1. Impute how many photon counts out of all the observed counts in each energy channel are attributed to the emission line of interest conditional on the current parameters. (I.e. Histogram)
2. Sample photon energies from a truncated normal (or t-) distribution.
3. Fit the new center of the emission line by treating the imputed photon energies as if observed.

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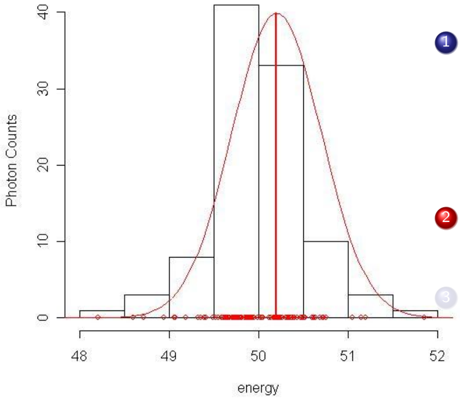

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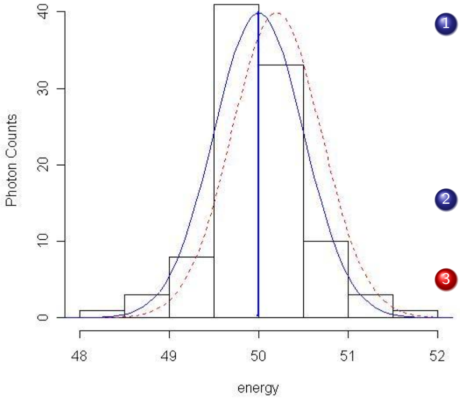

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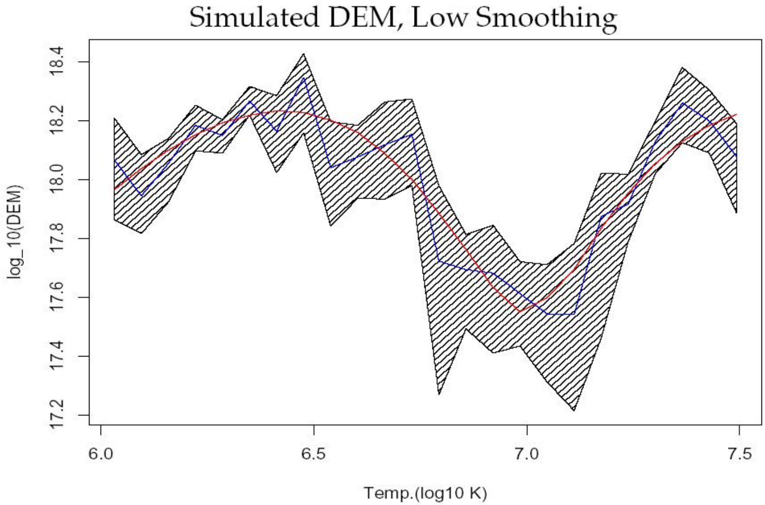
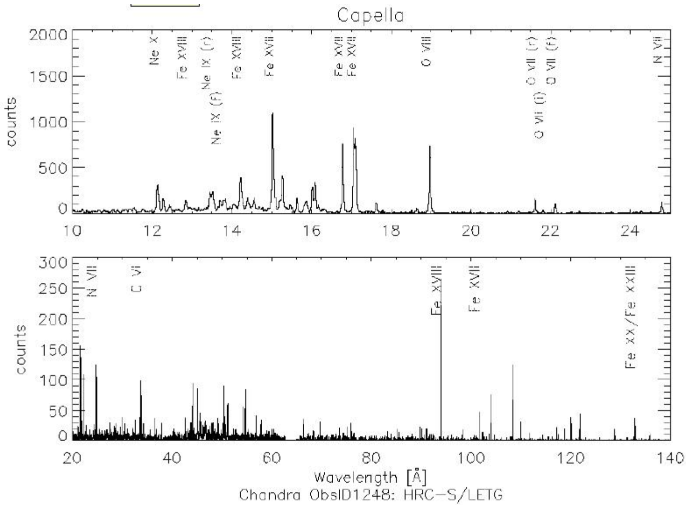
Simulation Results



Simulation Results

Element	Input Value	Mean	95% Interval
C	0.8	0.77	(0.70, 0.84)
Si	0.8	0.80	(0.74, 0.87)
N	2	2.00	(1.92, 2.10)
S	0.8	0.93	(0.75, 1.11)
O	0.5	0.50	(0.48, 0.52)
Ar	2.8	2.90	(2.68, 3.12)
Ne	5	5.06	(4.90, 5.22)
Ca	3.8	3.82	(3.45, 4.23)
Mg	3	2.99	(2.86, 3.12)
Fe	2	2.01	(1.95, 2.08)
Al	2.5	2.37	(1.57, 3.17)
Ni	2	2.03	(1.82, 2.26)

Capella Data - Chandra's HRC-S with LETGS



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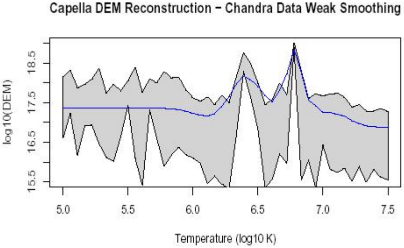
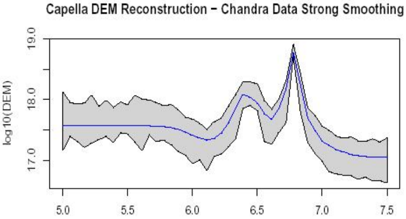
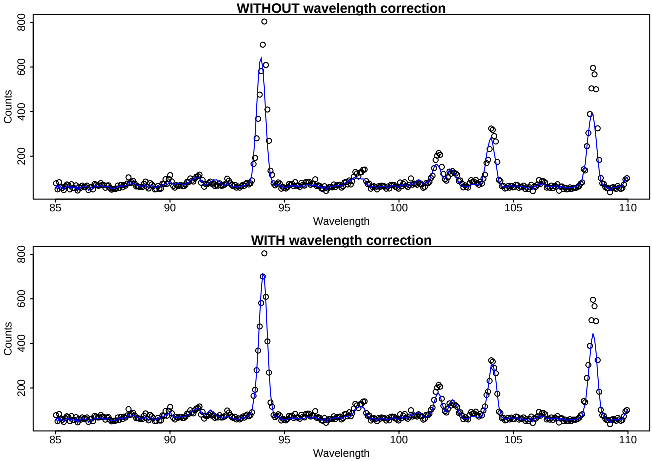




Capella Data - Chandra's HRC-S with LETGS

Element	Mode	Mean	95% Interval
C	0.155	0.149	(0.097, 0.205)
Si	0.266	0.255	(0.227, 0.286)
N	0.122	0.118	(0.110, 0.126)
S	0.300	0.293	(0.273, 0.315)
O	0.542	0.533	(0.492, 0.577)
Ar	0.235	0.251	(0.025, 0.555)
Ne	0.599	0.591	(0.540, 0.644)
Ca	0.362	0.356	(0.206, 0.517)
Mg	0.177	0.168	(0.085, 0.256)
Fe	0.303	0.295	(0.190, 0.405)
Al	0.428	0.422	(0.403, 0.442)
Ni	0.707	0.688	(0.616, 0.767)

Model Diagnostics - Atomic Error Correction



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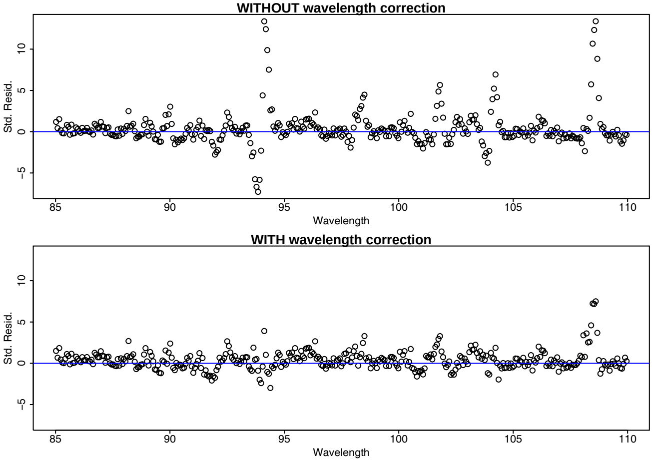
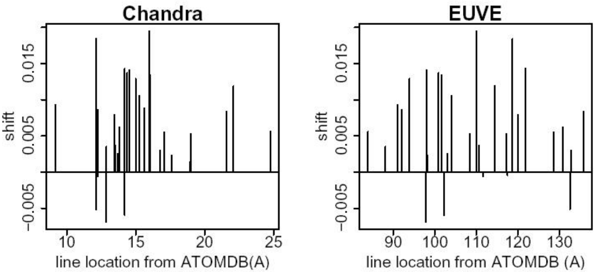


Emission Line Position Shift Due to Atomic Error



Model Diagnostics - Posterior Predictive Intervals